

Combinatorics (Problem Set)

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(*) might require calculus knowledge. (*) are past ISI problems

1. Let $\{x_n\}$ be a sequence such that $x_1 = 2, x_2 = 1$ and $2x_n - 3x_{n-1} + x_{n-2} = 0$ for $n > 2$. Find an expression for x_n .
2. In how many ways can a man wear 5 rings on the fingers (except the thumb) of one hand? (All the 5 rings should be utilized).
3. There are n straight lines in a plane, no two of which are parallel and no three are concurrent. Their point of intersections are joined. Prove that the number of fresh straight lines introduced is $\frac{n(n-1)(n-2)(n-3)}{8}$.
4. A closet has 5 pair of shoes. Find number of ways 4 shoes can be chosen from it so that there will be no complete pair.
5. Consider the smallest number in each of the $\binom{n}{r}$ subsets (of size r) of $S = \{1, 2, \dots, n\}$. Show that the arithmetic mean of the numbers so obtained is $\frac{n+1}{r+1}$.
6. On an island live 13 purple, 15 yellow and 17 maroon chameleons. When two chameleons of different colors meet, they both change into the third color. Is there a sequence of pairwise meetings after which all chameleons have the same color?
7. Prove by combinatorial argument : $\sum_{k=0}^n \binom{n}{k} k! k n^{n-k-1} = n^n$
8. Prove that the number of ways you can choose k numbers from $\{1, 2, \dots, n\}$ such that no two of the chosen numbers are consecutive is $\binom{n-k+1}{k}$
9. The Fibonacci sequence is defined by $a_1 = 1, a_2 = 1, a_{n+2} = a_{n+1} + a_n$. Find number of n for which $\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \dots + \frac{a_n}{2^n} > 2$.
10. "Run" is defined as a consecutive sequence of symbols till another symbol comes. For example, in AAAAABBBBAA, there are 2 runs of A and 1 run of B. In AAABBBBAAAAABBBBA, there are 3 runs of A and 2 runs of B. Now, assume there are 2 symbols A and B. R_1 = number of runs of A and R_2 = number of runs of B. Let $R = R_1 + R_2$ be the total number of runs. n_1 = total number of symbols A, n_2 = total number of symbols of B and $n = n_1 + n_2$. Let $R_{ij}, i = 1, 2, j = 1, 2, \dots, n_i$ denote respectively the number of runs of type i , which are of length j . So, $\sum_{j=1}^{n_i} j \cdot R_{ij} = n_i, \sum_{j=1}^{n_i} R_{ij} = R_i, i = 1, 2$ For example, in AABBAAB, $R_1 = R_2 = 2, n_1 = 4, n_2 = 3, n = 7, R = 4$. Now assume that you have observed a totally random sequence with n_1 many A and n_2 many B. Prove the followings:
 - (a) $r_1 = r_2$ or $r_1 = r_2 \pm 1$.
 - (b) $P(R_1 = r_1, R_2 = r_2) = \frac{c \binom{n_1-1}{r_1-1} \binom{n_2-1}{r_2-1}}{\binom{n_1+n_2}{n_1}}$, where $c = 2$ if $r_1 = r_2$ and $c = 1$ if $r_1 = r_2 \pm 1$.
 - (c) $P(R = 2d) = \frac{2 \binom{n_1-1}{d-1} \binom{n_2-1}{d-1}}{\binom{n_1+n_2}{n_1}}$
 - (d) $P(R = 2d + 1) = \frac{\binom{n_1-1}{d-1} \binom{n_2-1}{d} + \binom{n_1-1}{d} \binom{n_2-1}{d-1}}{\binom{n_1+n_2}{n_1}}$
 - (e) $P(R_1 = r_1) = \frac{\binom{n_1-1}{r_1-1} \binom{n_2+1}{r_1}}{\binom{n_1+n_2}{n_1}}$

- (f) Probability that the r_1 runs of n_1 objects of A and r_2 runs of n_2 objects of B consists of exactly $r_{1j}, j = 1, \dots, n_1$ and $r_{2j}, j = 1, \dots, n_2$ runs of length j respectively is

$$\frac{cr_1!r_2!}{\binom{n_1+n_2}{n_1} \prod_{i=1}^2 \prod_{j=1}^{n_i} r_{ij}!}$$

where $c = 2$ if $r_1 = r_2$ and $c = 1$ if $r_1 = r_2 \pm 1$.

(g) $P(R_{1j} = r_{ij}, j = 1, \dots, n_1) = \frac{\binom{n_2+1}{r_1} r_1!}{\binom{n_1+n_2}{n_1} \prod_{j=1}^{n_1} r_{ij}!}$

- (h) Now, let K = length of longest run of A. Take $n_1 = 5, n_2 = 6$. Find $P(K \leq 5)$.

- (i) In general,

$$P(K = k) = \sum_{r_1} \sum_{r_{11}, \dots, r_{1k}}^{r_{11} + \dots + r_{1k} = r_1} \frac{r_1! \binom{n_2+1}{r_1}}{\binom{n_1+n_2}{n_1} \prod_{j=1}^{n_1} r_{ij}!}$$

where, $\sum_{j=1}^k r_{ij} = r_1, \sum_{j=1}^k j r_{1j} = n_1, r_{1k} \geq 1, r_1 \leq n_1 - k + 1, r_1 \leq n_2 + 1$

11. Let $A_1, \dots, A_m, B_1, \dots, B_m$ be finite subsets of $\mathbb{N}$, such that $A_i \cap B_i = \emptyset, A_i \cap B_j \neq \emptyset$ for all $i \neq j$. Consider

$$\Omega = (\cup_{i=1}^m A_i) \cup (\cup_{i=1}^m B_i)$$

Now, let us consider a random permutation of Ω . Let E_i be the event that elements of A_i comes before elements of B_i . Now, prove the followings:

(a) $P(E_i) = \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}}$

- (b) Prove that atmost one of E_i can happen.

- (c) Hence prove that,

$$\sum_{i=1}^m \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1$$

- (d) If $|A_i| = k, |B_i| = l, m \leq \binom{k+l}{l}$

12. Let 18 equally spaced points be chosen on the circumference of a circle. How many ways can these points be connected in pairs such that the resulting 9 segments do not intersect?
13. 10 candidates participate for olympiad, which is organized around a table. There are 5 versions of the test and each candidate will receive exactly 1 version. No 2 consecutive candidates will get same version. How many ways there are to give the questions?
14. How many integral solutions are there to $x + y + z + t = 29$, when $x \geq 1, y > 1, z \geq 3, t \geq 0$?
15. If there are l objects of one kind, m objects of 2nd kind, n objects of third kind and so on, prove that the number of ways of choosing r objects from the $l + m + n + \dots$ objects is the coefficient of x^r in the expansion of $(1 + x + x^2 + \dots + x^l)(1 + x + \dots + x^m)(1 + x + \dots + x^n) \dots$
16. There are n straight lines in a plane such that n_1 of them are parallel in one direction, n_2 of them in different direction and so on, n_k are parallel in another direction such that $n_1 + \dots + n_k = n$. Also, no three of the given lines meet at a point. Prove that the total number of points of intersection is $\frac{1}{2}(n^2 - \sum_{r=1}^k n_r^2)$.
17. Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called full if it satisfies the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k - 1$, and moreover the first occurrence of $k - 1$ comes before the last occurrence of k . How many full sequences of length n are there?
18. For any finite set X , let $|X|$ denote the number of elements in X . Define

$$S_n = \sum |A \cap B|,$$

where the sum is taken over all ordered pairs (A, B) such that A and B are subsets of $\{1, 2, 3, \dots, n\}$ with $|A| = |B|$. For example, $S_2 = 4$ because the sum is taken over the pairs of subsets

$$(A, B) \in \{(\emptyset, \emptyset), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\})\},$$

giving $S_2 = 0 + 1 + 0 + 0 + 1 + 2 = 4$. Let $\frac{S_{2022}}{S_{2021}} = \frac{p}{q}$, where p and q are relatively prime positive integers. Find the remainder when $p + q$ is divided by 1000.

19. Consider the set $N_n = \{1, 2, \dots, n\}$. Let S be a subset of N_n such that $\forall A, B \in S$, we have $A \not\subseteq B$. Now prove the following steps.

(a) Let Π be a random permutation of N_n . Say, $S = \{A_1, \dots, A_m\}$. Let E_i be the event that $\{\Pi(1), \Pi(2), \dots, \Pi(|A_i|)\}$ is the **same set** as A_i . Prove that atmost one of E_i can happen out of $E_1, \dots, E_m$.

(b) Prove that, $P(E_i) = \frac{1}{\binom{n}{|A_i|}}$

(c) Combining (a) and (b) prove that

$$\sum_{A \in S} \frac{1}{\binom{n}{|A|}} \leq 1$$

20. In a sports tournament involving N teams, each team plays every other team exactly one. At the end of every match, the winning team gets 1 point and losing team gets 0 points. At the end of the tournament, the total points received by the individual teams are arranged in decreasing order as follows:

$$x_1 \geq x_2 \geq \dots \geq x_N.$$

Prove that for any $1 \leq k \leq N$,

$$\frac{N-k}{2} \leq x_k \leq N - \frac{k+1}{2}$$

21. Suppose $x_1, x_2, \dots, x_{2n}$ are distinct points in the plane. Prove that you can join the points in such a way such that there will be exactly n^2 many edges (a direct line between two points is denoted as an edge) with no triangle in the whole picture.
22. $n(> 1)$ lotus leaves are arranged in a circle. A frog jumps from a particular leaf from another under the following rule: It always moves clockwise. From starting it skips one leaf and then jumps to the next. After that it skips two leaves and jumps to the following. And the process continues. (Remember the frog might come back on a leaf twice or more.) Given that it reaches all leaves at least once. Show n cannot be odd.
23. There are 130 students in a class. A test is conducted which consists of 5 questions. Suppose the teacher first checks a copy of one student, and the student gets distinct marks on all of the questions (suppose i on the i th question, $i = 1, \dots, 5$). Suppose he gets $\{a, b, c, d, e\}$. Then, the teacher, being too lazy, takes a copy of a new student and randomly puts marks following the marks of the 1st student, also allowing repetition, i.e., for each question he/she can get anything within $\{a, b, c, d, e\}$ (i.e., $(1,1,1,1,1)$, $(1,5,5,5,5)$, etc are allowed.) Prove that there must exist two students who will get exact same set of marks. (i.e., $(1,2,1,4,5)$ and $(2,1,1,4,5)$ are considered as the same set.)
24. Assume A and B are playing a game. They have n distinct numbers. They are asked to produce sets of maximum length n out of those n many numbers. A chooses all possible d sets out of n without repetition. Call the total number of distinct sets $A(n, d)$. B chooses n many numbers out of the given numbers with repetition. Call the total number of distinct sets to be $B(n)$. Whoever gets more number of sets, win. For what value of $d(n)$, $A(n, d)$ is maximized. Find

$$\lim_{n \rightarrow \infty} \frac{A(n, d(n))}{B(n)}$$

if it exists. So, for large enough n , who will win, given both play optimally within their strategy? Assume that

$$\lim_{n \rightarrow \infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = 1$$

25. The circumference of a circle is divided into 45 arcs, each of length 1. Initially, there are 15 snakes, each of length 1, occupying every third arc. Every second, each snake independently moves either one arc left or one arc right, each with probability $\frac{1}{2}$. If two snakes ever touch, they merge to form a single snake occupying the arcs of both of the previous snakes, and the merged snake moves as one snake. Compute the expected number of seconds until there is only one snake left.
26. Compute the number of triples (f, g, h) of permutations on $\{1, 2, 3, 4, 5\}$ such that
- $$\begin{aligned} f(g(h(x))) &= h(g(f(x))) = g(x), \\ g(h(f(x))) &= f(h(g(x))) = h(x), \quad \text{and} \\ h(f(g(x))) &= g(f(h(x))) = f(x) \end{aligned}$$
- for all $x \in \{1, 2, 3, 4, 5\}$.
27. Svitlana writes the number 147 on a blackboard. Then, at any point, if the number on the blackboard is n , she can perform one of the following three operations:
- if n is even, she can replace n with $\frac{n}{2}$;
 - if n is odd, she can replace n with $\frac{n + 255}{2}$; and
 - if $n \geq 64$, she can replace n with $n - 64$.
28. Sets A , B , and C satisfy $|A| = 92$, $|B| = 35$, $|C| = 63$, $|A \cap B| = 16$, $|A \cap C| = 51$, $|B \cap C| = 19$. Compute the number of possible values of $|A \cap B \cap C|$.
29. Compute the number of ways to color 3 cells in a 3×3 grid so that no two colored cells share an edge.
30. Michel starts with the string HMMT. An operation consists of either replacing an occurrence of H with HM, replacing an occurrence of MM with MOM, or replacing an occurrence of T with MT. For example, the two strings that can be reached after one operation are HMMMT and HMOMT. Compute the number of distinct strings Michel can obtain after exactly 10 operations.
31. Let $\mathbb{N}$ denote the set of natural numbers, and let (a_i, b_i) , $1 \leq i \leq 9$, be nine distinct tuples in $\mathbb{N} \times \mathbb{N}$. Show that there are three distinct elements in the set $\{2^{a_i} 3^{b_i} : 1 \leq i \leq 9\}$ whose product is a perfect cube.
32. Compute the number of nonempty subsets $S \subseteq \{-10, -9, -8, \dots, 8, 9, 10\}$ that satisfy $|S| + \min(S) \max(S) = 0$.
33. Let n be a positive integer. A sequence of n positive integers (not necessarily distinct) is called full if it satisfies the following condition: for each positive integer $k \geq 2$, if the number k appears in the sequence then so does the number $k - 1$, and moreover the first occurrence of $k - 1$ comes before the last occurrence of k . How many full sequences of length n are there?